

# Numerical analysis of distributed optimal control problems governed by elliptic variational inequalities

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## Abstract

A continuous optimal control problem governed by an elliptic variational inequality was considered in Boukrouche-Tarzia, *Comput. Optim. Appl.*, 53 (2012), 375-392 where the control variable is the internal energy  $g$ . It was proved the existence and uniqueness of the optimal control and its associated state system. The objective of this work is to make the numerical analysis of the above optimal control problem, through the finite element method with Lagrange's triangles of type 1. We discretize the elliptic variational inequality which define the state system and the corresponding cost functional, and we prove that there exists a discrete optimal control and its associated discrete state system for each positive  $h$  (the parameter of the finite element method approximation). Finally, we show that the discrete optimal control and its associated state system converge to the continuous optimal control and its associated state system when the parameter  $h$  goes to zero.

*Key words:* Elliptic variational inequalities, distributed optimal control problems, numerical analysis, convergence of the optimal controls, free boundary problems.

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## 1 Introduction

We consider a bounded domain  $\Omega \subset \mathbb{R}^n$  whose regular boundary  $\partial\Omega = \Gamma_1 \cup \Gamma_2$  consists of the union of two disjoint portions  $\Gamma_1$  and  $\Gamma_2$  with  $\text{meas}(\Gamma_1) > 0$ . We consider the following free boundary problem (S):

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \quad \text{in} \quad \Omega; \quad (1.1)$$

$$u = b \quad \text{on} \quad \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \quad \text{on} \quad \Gamma_2; \quad (1.2)$$

where the function  $g$  in (1.1) can be considered as the internal energy in  $\Omega$ ,  $b$  is the constant temperature on  $\Gamma_1$  and  $q$  is the heat flux on  $\Gamma_2$ . The variational formulation of the above problem is given as: Find  $u = u_g \in K$  such that

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$$a(u, v - u) \geq (g, v - u)_H - \int_{\Gamma_2} q(v - u) \, ds, \quad \forall v \in K, \quad (1.3)$$

where

$$V = H^1(\Omega), \quad K = \{v \in V : v \geq 0 \text{ in } \Omega, v/\Gamma_1 = b\}, \quad V_0 = \{v \in V : v/\Gamma_1 = 0\},$$

$$H = L^2(\Omega), \quad Q = L^2(\Gamma_2), \quad (u, v)_Q = \int_{\Gamma_2} u v \, ds \quad \forall u, v \in Q,$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (u, v)_H = \int_{\Omega} u v \, dx \quad \forall u, v \in H.$$

We note that  $a$  is a bilinear, continuous, symmetric on  $V$  and a coercive form on  $V_0$  [39], that is to say: there exists a constant  $\lambda > 0$  such that

$$a(v, v) \geq \lambda \|v\|_V^2 \quad \forall v \in V_0. \quad (1.4)$$

In [11], the following continuous distributed optimal control problem associated with (S) or the elliptic variational inequality (1.3) was considered:

Problem (P): Find the continuous distributed optimal control  $g_{op} \in H$  such that

$$J(g_{op}) = \min_{g \in H} J(g) \quad (1.5)$$

where the quadratic cost functional  $J : H \rightarrow \mathbb{R}_0^+$  is defined by:

$$J(g) = \frac{1}{2} \|u_g\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (1.6)$$

with  $M > 0$  a given constant and  $u_g$  is the corresponding solution of the elliptic variational inequality (1.3) associated to the control  $g$ .

Several continuous optimal control problems are governed by elliptic variational inequalities, for example: the process of biological waste-water treatment; reorientation of a satellite by propellers; and economics: the problem of consumer regulation of a monopoly, etc. There exist an abundant literature for optimal control problems [4, 42, 50], for optimal control problems governed by elliptic variational equalities or inequalities [2, 3, 5, 6, 7, 8, 9, 11, 19, 20, 26, 28, 30, 32, 34, 38, 40, 45, 46, 52, 53, 54], for numerical analysis of variational inequalities or optimal control problems [10, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 27, 33, 35, 36, 37, 43, 47, 48, 49, 51], and for the numerical analysis of optimal control problems governed by an elliptic variational inequality there exist a few numbers of papers [1, 29, 31, 44].

The objective of this work is to make the numerical analysis of the optimal control problem (P) which is governed by the elliptic variational inequality (1.3) by proving the convergence of a discrete solution to the continuous optimal control problems.

In Section 2, we establish the discrete elliptic variational inequality (2.3) which is the discrete formulation of the continuous elliptic variational inequality (1.3), and we obtain that these discrete problems have unique solutions for all positive  $h$ . Moreover, on the adequate functional spaces these solutions are convergent when  $h \rightarrow 0^+$  to the solutions of the continuous elliptic variational inequality (1.3).

In Section 3, we define the discrete optimal control problem (3.2) corresponding to continuous optimal control problem (1.5). We prove the existence of a discrete solution for the optimal control problem  $(P_h)$  for each parameter  $h$  and we obtain the convergence of this family with its corresponding discrete state system to the continuous optimal control with the corresponding continuous state system of the problem  $(P)$ .

## 2 Discretization of the problem (S)

Let  $\Omega \subset \mathbb{R}^n$  a bounded polygonal domain;  $b$  a positive constant and  $\tau_h$  a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite elements of class  $C^0$  over  $\Omega$  being  $h$  the parameter of the finite element approximation which goes to zero [12, 18]. We take  $h$  equal to the longest side of the triangles  $T \in \tau_h$  and we can approximate the sets  $V$  and  $K$  by:

$$V_h = \{v_h \in C^0(\overline{\Omega}) : v_h|_T \in \mathbb{P}_1(T), \forall T \in \tau_h\}$$

$$V_{h0} = \{v_h \in C^0(\overline{\Omega}) : v_h|_{\Gamma_1} = 0; v_h|_T \in \mathbb{P}_1(T), \forall T \in \tau_h\}$$

and

$$K_h = \{v_h \in C^0(\overline{\Omega}) : v_h \geq 0, v_h|_{\Gamma_1} = b, v_h|_T \in \mathbb{P}_1(T) \forall T \in \tau_h\}$$

where  $\mathbb{P}_1(T)$  is the set of the polynomials of degree less than or equal to 1 in the triangle  $T$ . Let  $\Pi_h : V \rightarrow V_h$  be the corresponding linear interpolation operator and  $c_0 > 0$  a constant (independent of the parameter  $h$ ) such that [12]:

$$\|v - \Pi_h(v)\|_H \leq c_0 h^r \|v\|_r \quad \forall v \in H^r(\Omega), \quad 1 < r \leq 2 \quad (2.1)$$

$$\|v - \Pi_h(v)\|_V \leq c_0 h^{r-1} \|v\|_r \quad \forall v \in H^r(\Omega), \quad 1 < r \leq 2. \quad (2.2)$$

The discrete variational inequality formulation  $(S_h)$  of the system  $(S)$  is defined as: Find  $u_{hg} \in K_h$  such that

$$a(u_{hg}, v_h - u_{hg}) \geq (g, v_h - u_{hg})_H - \int_{\Gamma_2} q(v_h - u_{hg}) d\gamma, \quad \forall v_h \in K_h. \quad (2.3)$$

**Theorem 2.1.** *Let  $g \in H$ ,  $b > 0$  and  $q \in Q$  be, then there exist unique solution of the problem  $(S_h)$  given by the elliptic variational inequality (2.3).*

*Proof.* It follows from the application of Lax-Milgram Theorem [39, 41].  $\square$

**Lemma 2.1.** *Let  $g_1, g_2 \in H$ , and  $u_{hg_1}, u_{hg_2} \in K_h$  be the solutions of  $(S_h)$  for  $g_1$  and  $g_2$  respectively, then we have that:*

a) *there exist a constant  $C$  independent of  $h$  such that:*

$$\|u_{hg}\|_V \leq C, \quad \forall h > 0; \quad (2.4)$$

b)

$$\|u_{hg_2} - u_{hg_1}\|_V \leq \frac{1}{\lambda} \|g_2 - g_1\|_H \quad \forall h > 0; \quad (2.5)$$

c) if  $g_n \rightharpoonup g$  in  $H$  weak, then  $u_{hg_n} \rightarrow u_{hg}$  in  $V$  strong for each fixed  $h > 0$ .

*Proof.* a) If we consider  $v_h = b \in K_h$  in the discrete elliptic variational inequality (2.3) we have:

$$\begin{aligned} \lambda \|u_{hg} - b\|_V^2 &\leq a(u_{hg}, u_{hg} - b) \leq (g, u_{hg} - b)_H + (q, b - u_{hg})_Q \\ &\leq (\|g\|_H + \|q\|_Q \|\gamma_0\|) \|u_{hg} - b\|_V \end{aligned}$$

where  $\gamma_0$  is the trace operator and therefore (2.4) holds.

b) As  $u_{hg_1}$  and  $u_{hg_2}$  are respectively the solutions of discrete elliptic variational inequalities (2.3) for  $g_1$  y  $g_2$ , we have:

$$a(u_{hg_i}, v_h - u_{hg_i}) \geq (g_i, v_h - u_{hg_i})_H - (q, v_h - u_{hg_i})_Q, \quad \forall v_h \in K_h \quad (2.6)$$

for  $i = 1, 2$ . By coerciveness of  $a$  we deduce:

$$\begin{aligned} \lambda \|u_{hg_2} - u_{hg_1}\|_V^2 &\leq a(u_{hg_2} - u_{hg_1}, u_{hg_2} - u_{hg_1}) \leq (g_2 - g_1, u_{hg_2} - u_{hg_1})_H \\ &\leq \|g_2 - g_1\|_H \|u_{hg_2} - u_{hg_1}\|_V \quad \forall h > 0, \end{aligned}$$

thus (2.5) holds.

c) Let  $h > 0$  be. From item a) we have that  $\|u_{hg_n}\| \leq C \quad \forall n$ , then there exist  $\eta \in V$  such that  $u_{hg_n} \rightharpoonup \eta$  in  $V$  weak (in  $H$  strong). If we consider the discrete elliptic inequality (2.3) we have:

$$a(u_{hg_n}, v_h - u_{hg_n}) \geq (g_n, v_h - u_{hg_n})_H - (q, v_h - u_{hg_n})_Q$$

and using that  $a$  is a lower weak semi-continuous application then, when  $n$  goes to infinity, we obtain that:

$$a(\eta, v_h - \eta) \geq (g, v_h - \eta)_H - (q, v_h - \eta)_Q$$

and from uniqueness of the solution of problem  $(S_h)$ , we deduce that  $\eta = u_{hg} \in K_h$ .

Now, it is easily to see that:

$$a(u_{hg_n} - u_{hg}, u_{hg_n} - u_{hg}) \leq -(g - g_n, u_{hg_n} - u_{hg})_H$$

and from the coerciveness of  $a$  we obtain

$$\lambda \|u_{hg_n} - u_{hg}\|_V^2 \leq (g - g_n, u_{hg_n} - u_{hg})_H.$$

As  $u_{hg_n} \rightarrow u_{hg}$  in  $H$  and  $g_n \rightarrow g$  in  $H$ , by pass to the limit when  $n \rightarrow \infty$  in the previous inequality, we obtain

$$\lim_{n \rightarrow \infty} \|u_{hg_n} - u_g\|_V = 0.$$

□

Henceforth we will consider the following definitions [11]: Given  $\mu \in [0, 1]$  and  $g_1, g_2 \in H$ , we have the convex combinations of two data

$$g_3(\mu) = \mu g_1 + (1 - \mu)g_2 \in H, \quad (2.7)$$

the convex combination of two discrete solutions

$$u_{h3}(\mu) = \mu u_{hg_1} + (1 - \mu)u_{hg_2} \in K_h \quad (2.8)$$

and we define  $u_{h4}(\mu)$  as the associated state system which is the solution of the discrete elliptic variational inequality (2.3) for the control  $g_3(\mu)$ .

Then, we have the following properties:

**Lemma 2.2.** *Given the controls  $g_1, g_2 \in H$ , we have that:*

$$a) \quad \|u_{h3}\|_H^2 = \mu \|u_{hg_1}\|_H^2 + (1 - \mu) \|u_{hg_2}\|_H^2 - \mu(1 - \mu) \|u_{hg_2} - u_{hg_1}\|_H^2 \quad (2.9)$$

$$b) \quad \|g_3(\mu)\|_H^2 = \mu \|g_1\|_H^2 + (1 - \mu) \|g_2\|_H^2 - \mu(1 - \mu) \|g_2 - g_1\|_H^2 \quad (2.10)$$

*Proof.* a) From the definition (2.8) we get

$$\|u_{h3}\|_H^2 = \mu^2 \|u_{hg_1}\|_H^2 + (1 - \mu)^2 \|u_{hg_2}\|_H^2 + 2\mu(1 - \mu) (u_{hg_1}, u_{hg_2})_H$$

and

$$\|u_{hg_2} - u_{hg_1}\|_H^2 = \|u_{hg_2}\|_H^2 + \|u_{hg_1}\|_H^2 - 2(u_{hg_1}, u_{hg_2})_H,$$

then we conclude (2.9).

b) It follows from a similar method to the part a).  $\square$

**Theorem 2.2.** *If  $u_g$  and  $u_{hg}$  be the solutions of the elliptic variational inequalities (1.3) and (2.3) respectively for the control  $g \in H$ , then  $u_{hg}$  converge to  $u_g$  in  $V$  strong when  $h \rightarrow 0^+$ .*

*Proof.* From Lemma 2.1 we have that there exist a constant  $C > 0$  independent of  $h$  such that  $\|u_{hg}\|_V \leq C \quad \forall h > 0$ , then we conclude that there exists  $\eta \in V$  so that  $u_{hg} \rightharpoonup \eta$  in  $V$  weak as  $h \rightarrow 0^+$  and  $\eta \in K$ . On the other hand, given  $v \in K$  there exist  $v_h^*$  such that  $v_h^* \in K_h$  for each  $h$  and  $v_h^* \rightarrow v$  in  $V$  strong when  $h$  goes to zero. Now, by considering  $v_h^* \in K_h$  in the discrete elliptic variational inequality (2.3) we get:

$$a(u_{hg}, u_{hg}) \leq a(u_{hg}, v_h^*) - (g, v_h^* - u_{hg}) + (q, v_h^* - u_{hg})_Q \quad (2.11)$$

and when we pass to the limit as  $h \rightarrow 0^+$  in (2.11) by using that the bilinear form  $a$  is lower weak semicontinuous in  $V$  we obtain:

$$a(\eta, \eta) \leq a(\eta, v) - (g, v - \eta) + (q, v - \eta)_Q$$

that it is to say:

$$a(\eta, v - \eta) \geq (g, v - \eta) - (q, v - \eta)_Q \quad \forall v \in K$$

and, from the uniqueness of the solution of the discrete elliptic variational inequality (1.3), we obtain that  $\eta = u_g$ .

Now, we will prove the strong convergence. If we consider  $v = u_{hg} \in K_h \subset K$  in the elliptic variational inequality (1.3) and  $v_h = \Pi_h(u_g) \in K_h$  in (2.3), from the coerciveness of  $a$  and by some mathematical computation, we obtain that:

$$\begin{aligned} \lambda \|u_{hg} - u_g\|_V^2 &\leq a(u_{hg} - u_g, u_{hg} - u_g) \\ &\leq a(u_{hg}, \Pi_h(u_g) - u_g) - (g, \Pi_h(u_g) - u_g) + (q, \Pi_h(u_g) - u_g)_Q \end{aligned} \quad (2.12)$$

then by pass to the limit when  $h \rightarrow 0^+$  it results that  $\lim_{h \rightarrow 0^+} \|u_{hg} - u_g\|_V = 0$ .  $\square$

### 3 Discretization of the optimal control problem

Now, we consider the continuous optimal control problem which was established in (1.5). The associated discrete cost functional  $J_h : H \rightarrow \mathbb{R}_0^+$  is defined by the following expression:

$$J_h(g) = \frac{1}{2} \|u_{hg}\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (3.1)$$

and we establish the discrete optimal control problem  $(P_h)$  as: Find  $g_{op_h} \in H$  such that

$$J_h(g_{op_h}) = \min_{g \in H} J_h(g) \quad (3.2)$$

where  $u_{hg}$  is the associated state system solution of the problem  $(S_h)$  which was described for the discrete elliptic variational inequality (2.3) for a given control  $g \in H$ .

**Theorem 3.1.** *Given the control  $g \in H$ , we have:*

a)

$$\lim_{\|g\|_H \rightarrow \infty} J_h(g) = \infty.$$

b)  $J_h(g) \geq \frac{M}{2} \|g\|_H^2 - C \|g\|_H$  for some constant  $C$  independent of  $h$ .

c) The functional  $J_h$  is a lower weakly semi-continuous application in  $H$ .

d) There exists a solution of the discrete optimal control problem (3.2) for all  $h > 0$ .

*Proof.* a) From the definition of  $J_h(g)$  we obtain a) and b).

c) Let  $g_n \rightharpoonup g$  in  $H$  weak, then by using the equality  $\|g_n\|_H^2 = \|g_n - g\|_H^2 + \|g\|_H^2 + 2(g_n, g)_H$  we obtain that  $\|g\|_H \leq \liminf_{n \rightarrow \infty} \|g_n\|_H$ . Therefore, we have

$$\liminf_{n \rightarrow \infty} J_h(g_n) \geq \frac{1}{2} \|u_{hg}\|_H^2 + \frac{M}{2} \|g\|_H^2 = J_h(g).$$

d) It follows from [42].  $\square$

**Lemma 3.1.** *If the continuous state system has the regularity  $u_g \in H^r(\Omega)$  ( $1 < r \leq 2$ ) then we have the following estimations  $\forall g \in H$ :*

a)

$$\|u_{hg} - u_g\|_V \leq Ch^{\frac{r-1}{2}}, \quad (3.3)$$

b)

$$|J_h(g) - J(g)| \leq Ch^{\frac{r-1}{2}}. \quad (3.4)$$

where  $C$ 's are constants independents of  $h$ .

*Proof.* a) As  $u_g \in K$ , we have that  $\Pi_h(u_g) \in K_h \subset K$ . If we consider  $v_h = \Pi_h(u_g)$  in (2.3), by using the inequalities (2.12), we obtain:

$$\begin{aligned} \lambda \|u_{hg} - u_g\|_V^2 &\leq a(u_{hg} - u_g, u_{hg} - u_g) \\ &\leq a(u_{hg}, \Pi_h(u_g) - u_g) - (g, \Pi_h(u_g) - u_g) + \int_{\Gamma_2} q(\Pi_h(u_g) - u_g) d\gamma \\ &\leq C \|\Pi_h(u_g) - u_g\|_V \leq C \|u_g\|_r h^{r-1} \leq Ch^{r-1}, \end{aligned}$$

and then (3.3) holds.

b) From the definitions of  $J$  and  $J_h$ , it results:

$$J_h(g) - J(g) = \frac{1}{2} (\|u_{hg}\|_H^2 - \|u_g\|_H^2) = \frac{1}{2} [\|u_{hg} - u_g\|_H^2 + (u_g, u_{hg} - u_g)]$$

and therefore

$$|J_h(g) - J(g)| \leq \left(\frac{1}{2} \|u_{hg} - u_g\|_H + \|u_g\|_H\right) \|u_{hg} - u_g\|_H \leq Ch^{\frac{r-1}{2}}.$$

□

Following the idea given in [11] we define an open problem: Given the controls  $g_1, g_2 \in H$ ,

$$0 \leq u_{h4}(\mu) \leq u_{h3}(\mu) \text{ in } \Omega, \quad \forall \mu \in [0, 1], \forall h > 0, \quad (3.5)$$

or

$$\|u_{h4}(\mu)\|_H \leq \|u_{h3}(\mu)\|_H \quad \forall \mu \in [0, 1], \forall h > 0. \quad (3.6)$$

**Remark 1:** We have that (3.5)  $\Rightarrow$  (3.6).

**Remark 2:** The equivalent inequality (3.5) for the continuous optimal control problem ( $P$ ) is true, that is [11]: for all  $g_1, g_2 \in H$ ,

$$0 \leq u_4(\mu) \leq u_3(\mu) \text{ in } \Omega, \quad \forall \mu \in [0, 1]. \quad (3.7)$$

where  $u_3(\mu) = \mu u_{g_1} + (1 - \mu) u_{g_2} \in K$ ,  $u_{g_i}$  ( $i = 1, 2$ ) is the unique solution of the elliptic variational inequality (1.3) when we consider  $g_i$  instead of  $g$ , and  $u_4(\mu)$  is the unique solution of the elliptic variational inequality (1.3) when we consider  $g_3(\mu)$  instead of  $g$ .

**Remark 3:** If (3.6) (or (3.5)) is true, then the functional  $J_h$  is H-elliptic and a strictly convex application because we have

$$\begin{aligned} &\mu J_h(g_1) + (1 - \mu) J_h(g_2) - J_h(g_3(\mu)) \\ &= \frac{\mu(1 - \mu)}{2} \|u_{hg_2} - u_{hg_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu) \|g_2 - g_1\|_H^2 + \frac{1}{2} [\|u_{h3}\|_H^2 - \|u_{h4}\|_H^2] \\ &\geq \frac{\mu(1 - \mu)}{2} \|u_{hg_2} - u_{hg_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu) \|g_2 - g_1\|_H^2 > 0 \end{aligned}$$

and therefore, the uniqueness for the discrete optimal control problem ( $P_h$ ) in the theorem 3.1 holds.

**Theorem 3.2.** *Let  $u_{g_{op}} \in K$  be the continuous state system associated to the optimal control  $g_{op} \in H$  which is the solution of the continuous distributed optimal control problem (1.5). If, for each  $h > 0$ , we choose an optimal control  $g_{op_h} \in H$  which is the solution of the discrete distributed optimal control problem (3.2) and its corresponding discrete state system  $u_{h g_{op_h}} \in K_h$ , we obtain that:*

$$u_{h g_{op_h}} \rightarrow u_{g_{op}} \quad \text{on } V \text{ strong and } g_{op_h} \rightarrow g_{op} \quad \text{on } H \text{ strong when } h \rightarrow 0^+.$$

*Proof.* Let be  $h > 0$  and  $g_{op_h}$  a solution of (3.2), and  $u_{h g_{op_h}}$  its associated discrete optimal state system which is the solution of the discrete elliptic variational inequality (2.3) for each  $h > 0$ . From (3.1) we have that for all  $g \in H$

$$J_h(g_{op_h}) = \frac{1}{2} \|u_{h g_{op_h}}\|_H^2 + \frac{M}{2} \|g_{op_h}\|_H^2 \leq \frac{1}{2} \|u_{hg}\|_H^2 + \frac{M}{2} \|g\|_H^2.$$

Then, if we consider  $g = 0$  and  $u_{h0}$  his corresponding associated state system, it results that:

$$J_h(g_{op_h}) = \frac{1}{2} \|u_{h g_{op_h}}\|_H^2 + \frac{M}{2} \|g_{op_h}\|_H^2 \leq \frac{1}{2} \|u_{h0}\|_H^2.$$

From the Lemma 2.1 we have that  $\|u_{h0}\|_H \leq C \quad \forall h$ , then we can obtain:

$$\|u_{h g_{op_h}}\|_H \leq C \quad \forall h > 0 \quad (3.8)$$

and

$$\|g_{op_h}\|_H \leq \frac{1}{M} \|u_{h0}\|_H \leq \frac{1}{M} C \quad \forall h > 0. \quad (3.9)$$

If we consider  $v_h = b \in K_h$  in the inequality (2.3) for  $g_{op_h}$ , we obtain:

$$a(u_{h g_{op_h}}, b - u_{h g_{op_h}}) \geq (g_{op_h}, b - u_{h g_{op_h}}) - (q, b - u_{h g_{op_h}})_Q, \quad (3.10)$$

therefore:

$$a(u_{h g_{op_h}} - b, u_{h g_{op_h}} - b) \leq (g_{op_h}, u_{h g_{op_h}} - b) - (q, u_{h g_{op_h}} - b)_Q, \quad (3.11)$$

and from the coerciveness of the application  $a$  we have that  $\|u_{h g_{op_h}} - b\|_V \leq C$  and in consequence  $\|u_{h g_{op_h}}\|_V \leq C$ .

Now we can say that there exist  $\eta \in V$  and  $f \in H$  such that  $u_{h g_{op_h}} \rightharpoonup \eta$  in  $V$  weak (in  $H$  strong), and  $g_{op_h} \rightharpoonup f$  in  $H$  weak when  $h \rightarrow 0^+$ . Then,  $\eta/\Gamma_1 = b$  and  $\eta \geq 0$  in  $\Omega$  i.e.,  $\eta \in K$ .

Let given  $v \in K$ , there exist  $v_h \in K_h$  such that  $v_h \rightarrow v$  in  $V$  strong when  $h \rightarrow 0^+$ . Then, if we consider the variational elliptic inequality (2.3) for  $g = g_{op_h}$  we have:

$$a(u_{h g_{op_h}}, v_h) \geq a(u_{h g_{op_h}}, u_{h g_{op_h}}) + (g_{op_h}, v_h - u_{h g_{op_h}}) - (q, v_h - u_{h g_{op_h}})_Q. \quad (3.12)$$

Taking into account that the application  $a$  is a lower weak semi-continuous application in  $V$  and by pass to the limit when  $h$  goes to zero in (3.10) we obtain that:

$$a(\eta, v - \eta) \geq (f, v - \eta) - (q, v - \eta)_Q, \quad \forall v \in K$$



and by the uniqueness of the solution of the problem given by the elliptic variational inequality (1.3), we deduce that  $\eta = u_f$ .

Finally, the norm on  $H$  is a lower semi-continuous application in the weak topology, then we can prove that:

$$\begin{aligned} J(f) &= \frac{1}{2}\|u_f\|_H^2 + \frac{M}{2}\|f\|_H^2 \leq \liminf_{h \rightarrow 0^+} J_h(g_{op_h}) \leq \liminf_{h \rightarrow 0^+} J_h(g) = \frac{1}{2} \lim_{h \rightarrow 0^+} \|u_{hg}\|_H^2 + \frac{M}{2}\|g\|_H^2 \\ &= \frac{1}{2}\|u_g\|_H^2 + \frac{M}{2}\|g\|_H^2 = J(g), \quad \forall g \in H \end{aligned}$$

and because the uniqueness of the optimal problem (1.5), it results that  $f = g_{op}$  and  $\eta = u_{g_{op}}$ .

Now, if we consider  $v = u_{hg_{op_h}} \in K_h \subset K$  in the elliptic variational inequality (1.3) for the control  $g_{op}$  and we define  $z_h = u_{hg_{op_h}} - u_{g_{op}}$ , we have that:

$$a(z_h, z_h) \leq a(u_{hg_{op_h}}, u_{hg_{op_h}}) - a(u_{hg_{op_h}}, u_{g_{op}}) - (g_{op}, u_{hg_{op_h}} - u_{g_{op}})_Q + (q, u_{hg_{op_h}} - u_{g_{op}})_Q,$$

and by consider  $v = \Pi_h(u_{g_{op}}) \in K_h$  for  $g = g_{op_h}$  in the inequality (2.3) we obtain:

$$a(u_{hg_{op_h}}, u_{hg_{op_h}}) \leq -(g_{op_h}, \Pi_h(u_{g_{op}}) - u_{hg_{op_h}})_Q + (q, \Pi_h(u_{g_{op}}) - u_{hg_{op_h}})_Q + a(u_{hg_{op_h}}, \Pi_h(u_{g_{op}})).$$

and then by the coerciveness of  $a$  we get

$$\begin{aligned} \lambda \|z_h\|_V^2 &\leq (q, \Pi_h(u_{g_{op}}) - u_{g_{op}})_Q + a(u_{hg_{op_h}}, \Pi_h(u_{g_{op}}) - u_{g_{op}}) \\ &\quad + (g_{op_h} - g_{op}, u_{hg_{op_h}} - u_{g_{op}}) - (g_{op}, \Pi_h(u_{g_{op}}) - u_{g_{op}}) \end{aligned} \quad (3.13)$$

When we pass to the limit as  $h \rightarrow 0$  in (3.11) and by using the strong convergence of  $u_{hg_{op_h}}$  to  $u_{g_{op}}$  on  $H$  and the weak convergence of  $g_{op_h}$  to  $g_{op}$  on  $H$ , we have:

$$\lim_{h \rightarrow 0^+} \|u_{g_{op}} - u_{hg_{op_h}}\|_V = 0. \quad (3.14)$$

The strong convergence of the optimal controls  $g_{op_h}$  to  $g_{op}$  is obtained by using Theorem 3.1 and  $g_{op_h} \rightharpoonup g_{op}$  weakly on  $H$ , i.e.

$$J(g_{op}) = \frac{1}{2}\|u_{g_{op}}\|_H^2 + \frac{M}{2}\|g_{op}\|_H^2 \leq \liminf_{h \rightarrow 0^+} J_h(g_{op_h})$$

$$\leq \liminf_{h \rightarrow 0^+} J_h(g_{op}) = \liminf_{h \rightarrow 0^+} \frac{1}{2}\|u_{g_{op}}\|_H^2 + \frac{M}{2}\|g_{op}\|_H^2 = J(g_{op},)$$

then  $\lim_{h \rightarrow 0} \|g_{op_h}\|_H = \|g_{op}\|_H$  and therefore  $\lim_{h \rightarrow 0^+} \|g_{op_h} - g_{op}\|_H = 0$ .

□

## 4 Conclusions

We have proved the convergence of a discrete optimal control and its corresponding discrete state system governed by a discrete elliptic variational inequality to the continuous optimal control and its corresponding continuous state system which is also governed by a continuous elliptic variational inequality by using the finite element method with Lagrange's triangles of type 1. Moreover, it is an open problem to obtain the error estimates as a function of the parameter  $h$  of the finite element method.

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